THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VII. JUNE-JULY, 1900. No. 6-7.

ON A DETERMINANT EACH OF WHOSE ELEMENTS IS THE PRODUCT OF k FACTORS.

By PROF. W. H. METZLER, Syracuse University, Syracuse, N. Y.

1. In Muir's *Theory of Determinants*, page 117, the following example (slightly modified) is given without comment:

$$\begin{vmatrix} h_1m_1 & h_1n_1 & k_1x_1 & k_1z_1 \\ h_2m_1 & h_2n_1 & k_2x_1 & k_2z_1 \\ p_1m_2 & p_1n_2 & q_1x_2 & q_1z_2 \\ p_2m_2 & p_2n_2 & q_2x_2 & q_2z_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 & 0 & 0 \\ h_2 & k_2 & 0 & 0 \\ 0 & 0 & p_1 & q_1 \\ 0 & 0 & p_2 & q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & 0 & m_2 & 0 \\ n_1 & 0 & n_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & z_1 & 0 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} h_1 k_1 \\ h_2 k_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 q_1 \\ p_2 q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 n_1 \\ m_2 n_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix} .$$

From this it is seen that if we take n_1 determinants

$$| b_{1n_2}^{(n_2, j_1)} | , \; (j_1=1,\ 2 \ldots . n_1),$$

of order n_2 and form a determinant of order $n=n_1n_2$ by arranging them along the principal diagonal as in the first factor of the right-hand member of the first equation in the above example, all the other elements being zeros, and n_2 determinants

$$| b_{1n_1}^{(n_1, j_2)} | , \; (j_2=1,\ 2 \ldots . n_2),$$

of order n_1 and form another determinant of order n in which the elements in the $(\alpha n_2+\beta)$th column of the βth n_1 rows are the elements of the $(\alpha+1)$th column of

$$| b_{1n_1}^{(n_1,\beta)} |, (\alpha=0, 1 \ldots n-1), (\beta=1, 2 \ldots n_2),$$

all the other elements being zeros, the product of these two determinants of the nth order is a determinant of the nth order whose elements are products of two factors.

It is also seen that the elements of $| b_{1n_2}^{(n_2,j_1)} |$ are found in all the columns, but in the j_1th n_2 rows only of the product, and that the elements of $| b_{1n_1}^{(n_1,j_2)} |$ are found in all the rows, but in the (αn_2+j_2)th columns only of the product.

2. We may now form two determinants of the nth order $(n=n_1n_2n_3)$ precisely as in Art. 1, first by taking n_3 determinants of order n_1n_2 each of whose elements is the product of two factors as there found, and second by taking n_1n_2 determinants

$$| b_{1n_3}^{(n_3,j_3)} |, (j_3=1, 2 \ldots (n/n_3)$$

of order n_3, and the product of these two determinants of the nth order will be a determinant of the nth order each of whose elements is the product of three factors, one factor an element from a determinant of each of the three orders.

Continuing this process we arrive at a determinant of the order $n=n_1n_2 \ldots n_k$, each of whose constituents is the product of k factors, one factor an element from a determinant of each of the k given orders $n_1, n_2 \ldots n_k$

3. Let $\left\{ \frac{\alpha}{\beta} \right\}$ denote the greatest integer in $\frac{\alpha}{\beta}$, $R\frac{\alpha}{\beta}$ denote the remainder on dividing α by β,

$$| b_{1n_g}^{(n_g,j_g)} |, (g=1, 2 \ldots k), (j_g=1, 2 \ldots n/n_g),$$

denote the n/n_g determinants of order n_g. Then if $n=n_1n_2 \ldots n_k$ we have the theorem

$$A= | a_{1n} | =\Pi | b_{1n_g}^{(n_g,j_g)} |, (g=1, 2 \ldots k), (j_g=1, 2 \ldots n/n_g) \ldots (1).$$

Where the element in the xth row and yth column of A,

$$a_{xy}=b^{\left(n_1, \left\{\frac{x-1}{n_1}\right\}+1\right)}_{R\frac{x-1}{n_1}+1, \left\{\frac{y-1}{n/n_1}\right\}+1}$$

$$\prod_{h=2}^{h=k} b\left(n_h,\ \left\{\frac{x-1}{n_1 n_2 \ldots n_h}\right\} n_1 n_2 \ldots n_{h-1} + \left\{\frac{\frac{y-1}{n}}{n_1 n_2 \ldots n_{h-1}}\right\} + 1\right) \ldots (2).$$
$$\left\{\frac{R\frac{\frac{y-1}{n}}{n_1 n_2 \ldots n_{h-1}}}{\frac{n}{n_1 n_2 \ldots n_h}}\right\} + 1,\ \left\{\frac{R\frac{x-1}{n_1 n_2 \ldots n_h}}{n_1 n_2 \ldots n_{h-1}}\right\} + 1$$

Each of the elements $b_{\alpha_g\beta_g}{}^{(n_g,\ i_g)}$ $(\alpha_g,\ \beta_g = 1,\ 2 \ldots n_g)$ occurs in $\frac{n}{n_g \ldots n_k}$ rows and in $\frac{n}{n_1 \ldots n_g}$ columns of A.

If we write $i_g = \lambda_g . n_1 n_1 \ldots n_{g-1} + \mu_g$, where

$$\left\{\lambda_g = 0,\ 1 \ldots \left(\frac{n}{n_1 \ldots n_g} - 1\right)\right\},$$

$$\{\mu_g = 1,\ 2 \ldots (n_1,\ n_2 \ldots n_{g-1})\},$$

and understand that when $g=1$, $n_1 n_2 \ldots n_{g-1} = 1$.

Then the element $b_{\alpha_g\beta_g}{}^{(n_g,\ \lambda_g n_1 \ldots n_{g-1} + \mu_g)}$ occurs in the $(\lambda_g\ n_g + \alpha_g)$th $n_1 n_2 \ldots n_{g-1}$ rows, and in the $\{(\mu_g - 1)n_g + \beta_g\}$th $\frac{n}{n_1 \ldots n_g}$ columns of A.

4. If all the determinants of the same order n_g, $(g=1,\ 2 \ldots k)$ are equal the theorem becomes

$$A = \Pi \mid b_{1n_g}{}^{(n_g)} \mid^{ig},\ ig = n/n_g \quad \ldots (3).$$

If $n_1 = n_2 = n_3 = \ldots = n_k = m$, then $n = m^k$ and the theorem takes the form

$$A = \Pi \mid b_{1m}{}^{(h)} \mid,\ (h = 1,\ 2 \ldots k.m^{k-1}) \quad \ldots (4).$$

If all the determinants are of the same order and equal to each other then the theorem becomes

$$A = \mid b_{1m} \mid^{k.m^{k-1}} \quad \ldots (5).$$

After finding the general theorem in Art. 3, the special cases (3) and (5) were first made known to me by Prof. E. H. Moore, who discovered them before knowing of the general theorem, and who immediately made the generalization, using the different notation, on receiving the Muir reference.

December 5, 1898.